Integrability of cyclotron motion

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Effects of the polarization of the electric field upon the integrability of cyclotron motion are studied. It is shown that relativistic cyclotron motion is integrable when the electric field is circularly polarized, but is not when the electric field is linearly or elliptically polarized. The motion in nonrelativistic limit is regular regardless of the polarization of the electric field. [S1063-651X(97)08809-0]

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I. INTRODUCTION

The dynamics of charged particles moving in electric and magnetic fields is of great importance in accelerator physics, plasma physics, and atomic and optical physics [1,2]. The question concerning the integrability of the motion of such particles is not just of academic interest, because it is directly related to the performance of such practical devices as the cyclotron, tokamak, free-electron laser, and Penning trap. Recently, it has been found that some interesting nonlinear effects such as chaos and bistable hysterisis are exhibited by electrons in these devices [3–8], especially when relativistic effects play a non-negligible role [9].

In this paper, we study the dynamics of relativistic and nonrelativistic cyclotron motion in a mutually orthogonal uniform magnetic field and an oscillating electric field. We address, in particular, the question of the integrability of cyclotron motion in relation to the polarization of the electric field. This study is motivated by two recent theoretical reports; Kim and Lee [10] have shown that chaotic cyclotron motion occurs when the electric field is linearly polarized, while Bourdier *et al.* [11] have shown that the cyclotron motion is integrable when the electric field is circularly polarized.

Theoretical analysis of the cyclotron motion can often be carried out most effectively if a canonical transformation to an appropriate rotating frame of reference is performed. For a discussion of nonrelativistic cyclotron motion, it helps greatly to look at the motion in the frame rotating at the Larmor frequency. No such convenient frame exists when the motion becomes relativistic, except when the motion occurs in a circularly polarized electric field, in which case an analysis can be performed conveniently in the frame rotating with the electric field. Our proof of the integrability of the motion for the case of a circularly polarized electric field is thus given based on an analysis performed in this rotating frame and seems to be simpler and physically more transparent than the proof given by Bourdier *et al.* [11].

II. THEORY

We consider a particle of mass m and charge q moving under the influence of a uniform magnetic field $\vec{B}=B_0\hat{e}_z$ and a time-periodic electric field $\vec{E}$ assumed to be orthogonal to $\vec{B}$. From here on the vector $\hat{e}_i$ denotes a unit vector along the i direction. We assume that the initial velocity has no z component and consider only a planar motion in the xy plane. The electric field in an arbitrary elliptically polarized state can be written as

$$\vec{E}=E_0\cos wt\,\hat{e}_x+E_0\cos(wt+\phi)\hat{e}_y, \tag{1}$$

where $\phi=0$ or $\pm\pi$ corresponds to linear polarization and $\phi=\pm\pi/2$ to circular polarization. The scalar and vector potentials φ and $\vec{A}$ can be taken to be

$$\varphi=-xE_0\cos wt-yE_0\cos(wt+\phi), \tag{2}$$

$$\vec{A}=-\frac{yB_0}{2}\hat{e}_x+\frac{xB_0}{2}\hat{e}_y. \tag{3}$$

Here and throughout the paper Gaussian units are used.

A. Nonrelativistic cyclotron motion

Let us first consider the motion in the nonrelativistic limit. The Lagrangian is given by

$$L=\frac{1}{2}m(\dot{x}^2+\dot{y}^2)+\frac{qB_0}{2c}(x\dot{y}-y\dot{x})+qE_0x\cos wt + qE_0y\cos(wt+\phi). \tag{4}$$

Equation (4) can be written, in terms of cylindrical coordinates, as

$$L=\frac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2)+\frac{qB_0}{2c}r^2\dot{\theta}+qE_0r\cos\theta\cos wt + qE_0r\sin\theta\cos(wt+\phi). \tag{5}$$

In the frame rotating (clockwise if $q>0$) at the Larmor frequency

$$\Omega=\frac{qB_0}{2mc}, \tag{6}$$

the Lagrangian takes a simple form and is given by

$$L=\frac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2)-\frac{1}{2}m\Omega^2r^2+qE_0r\cos(\theta-\Omega t)\cos wt+qE_0r\sin(\theta-\Omega t)\cos(wt+\phi), \tag{7}$$

where the primes indicating the rotating frame have been dropped. Returning to Cartesian coordinates, Eq. (7) becomes

$$L=\frac{1}{2}m(\dot{x}^2+\dot{y}^2)-\frac{1}{2}m\Omega^2(x^2+y^2)+qE_0x[\cos\Omega t\cos wt-\sin\Omega t\cos(wt+\phi)]+qE_0y[\sin\Omega t\cos wt+\cos\Omega t\cos(wt+\phi)], \tag{8}$$

which immediately yields the equations of motion

$$m\ddot{x}+m\Omega^2x=qE_0[\cos\Omega t\cos wt-\sin\Omega t\cos(wt+\phi)], \tag{9}$$

$$m\ddot{y}+m\Omega^2y=qE_0[\sin\Omega t\cos wt+\cos\Omega t\cos(wt+\phi)]. \tag{10}$$

Equations (9) and (10) indicate that, when viewed in the frame rotating at the Larmor frequency, the motion being considered appears as driven harmonic oscillations in the x and y directions, respectively, with the frequency of the driving force given by $w\pm\Omega$. The motion viewed in this rotating frame is thus regular regardless of the polarization of the electric field and so is the motion in the laboratory frame.

B. Relativistic cyclotron motion

We now wish to consider the case when the motion is relativistic. What complicates the matter in the relativistic case is the fact that the angular frequency of a charged particle in the presence of a uniform magnetic field B_0 is given by

$$\Omega=\frac{qB_0}{2\gamma mc} \tag{11}$$

and is no longer constant; it depends on the velocity of the particle through the quantity $\gamma=1/\sqrt{1-v^2/c^2}$. The frame of reference rotating at a constant Larmor frequency no longer exists. If, however, the electric field is circularly polarized, one can choose to view the motion in the frame rotating at the frequency of the field. In this frame, the electric field will appear as a stationary field and one might hope to get a simple picture of the motion at least for the case of a circularly polarized electric field.

With the above argument in mind, we proceed to consider relativistic cyclotron motion when the field is circularly polarized, i.e., when

$$\vec{E}=E_0\cos wt\hat{e}_x+E_0\sin wt\hat{e}_y. \tag{12}$$

The Lagrangian can be written as

$$L=-mc^2\sqrt{1-\frac{\dot{x}^2+\dot{y}^2}{c^2}}+\frac{qB_0}{2c}(x\dot{y}-y\dot{x})+qE_0x\cos wt+qE_0y\sin wt. \tag{13}$$

Transforming to the frame rotating counterclockwise at frequency w, we have

$$L=-mc^2\sqrt{1-\frac{1}{c^2}[\dot{x}^2+\dot{y}^2+w^2(x^2+y^2)+2w(x\dot{y}-y\dot{x})]}+\frac{qB_0}{2c}w(x^2+y^2)+\frac{qB_0}{2c}(x\dot{y}-y\dot{x})+qE_0x, \tag{14}$$

where again the primes have been dropped. The generalized momenta can immediately be obtained as

$$p_x=\frac{m\dot{x}-mwy}{\sqrt{1-\frac{1}{c^2}[\dot{x}^2+\dot{y}^2+w^2(x^2+y^2)+2w(x\dot{y}-y\dot{x})]}}-\frac{qB_0}{2c}y, \tag{15}$$

$$p_y=\frac{m\dot{y}+mwx}{\sqrt{1-\frac{1}{c^2}[\dot{x}^2+\dot{y}^2+w^2(x^2+y^2)+2w(x\dot{y}-y\dot{x})]}}+\frac{qB_0}{2c}x \tag{16}$$

and the Hamiltonian as

$$H=\sqrt{m^2c^4+\left(p_x+\frac{qB_0}{2c}y\right)^2c^2+\left(p_y-\frac{qB_0}{2c}x\right)^2c^2}-w(xp_y-yp_x)-qE_0x. \tag{17}$$

One notes from Eq. (17) that, in this rotating frame, the Hamiltonian is time independent and thus is a constant of motion.

In order to find the second constant of motion, we first write Hamilton's equations of motion

$$\frac{dx}{dt}=\frac{\left(p_x+\frac{qB_0}{2c}y\right)c^2}{A}+wy, \tag{18}$$

$$\frac{dy}{dt}=\frac{\left(p_y-\frac{qB_0}{2c}x\right)c^2}{A}-wx, \tag{19}$$

$$\frac{dp_x}{dt}=\frac{\left(p_y-\frac{qB_0}{2c}x\right)\frac{qB_0c}{2}}{A}+wp_y+qE_0, \tag{20}$$

$$\frac{dp_y}{dt}=-\frac{\left(p_x+\frac{qB_0}{2c}y\right)\frac{qB_0c}{2}}{A}-wp_x, \tag{21}$$

where

$$A=A(x,y,p_x,p_y)$$
$$=\sqrt{m^2c^4+\left(p_x+\frac{qB_0}{2c}y\right)^2c^2+\left(p_y-\frac{qB_0}{2c}x\right)^2c^2}. \tag{22}$$

From Eqs. (18)–(21) we obtain

$$\frac{d}{dt}\left(p_x-\frac{qB_0}{2c}y\right)=w\left(p_y+\frac{qB_0}{2c}x\right)+qE_0, \tag{23}$$

$$\frac{d}{dt}\left(p_y+\frac{qB_0}{2c}x\right)=-w\left(p_x-\frac{qB_0}{2c}y\right). \tag{24}$$

Equations (23) and (24) immediately yield

$$\frac{1}{2}\frac{d}{dt}\left[\left(p_x-\frac{qB_0}{2c}y\right)^2+\left(p_y+\frac{qB_0}{2c}x\right)^2\right]+\frac{qE_0}{w}\frac{d}{dt}\left(p_y+\frac{qB_0}{2c}x\right)=0. \tag{25}$$

The constant of motion can thus be taken as

$$C=\frac{1}{2m}\left[\left(p_x-\frac{qB_0}{2c}y\right)^2+\left(p_y+\frac{qB_0}{2c}x\right)^2\right]+\frac{qE_0}{mw}\left(p_y+\frac{qB_0}{2c}x\right). \tag{26}$$

It can be shown by straightforward algebra that C and H are in involution, i.e., that the Poisson bracket $[C,H]$ vanishes. We can thus conclude that relativistic cyclotron motion for the case of a circularly polarized electric field is integrable in this rotating frame and so is the motion in the laboratory frame. It should be mentioned here that the quantity C and the nonrelativistic version of H given by

$$H=\frac{1}{2m}\left(p_x+\frac{qB_0}{2c}y\right)^2+\frac{1}{2m}\left(p_y-\frac{qB_0}{2c}x\right)^2-w(xp_y-yp_x)-qE_0x \tag{27}$$

are also constants of motion for the case when the cyclotron motion is nonrelativistic and the electric field is circularly polarized.

It should be noted that only a circularly polarized electric field appears as stationary in the frame rotating at frequency w. For all other polarizations, the Hamiltonian in this rotating frame is still time dependent and apparently the particle motion is nonintegrable as long as relativistic effects are taken into account. It has already been shown through numerical computation that the particle can display chaotic behavior when the electric field is linearly polarized [10]. We have also performed numerical computation for the case when the electric field is elliptically polarized and observed chaotic behavior. The threshold field at which the onset of chaos occurs increases as the angle ϕ is increased from zero (linear polarization) toward $\pi/2$ (circular polarization). Further details will be described elsewhere.

C. Cyclotron motion in two electric fields

One may also ask about the integrability of the cyclotron motion when a second oscillating electric field, in addition to the uniform magnetic field and oscillating electric field, is present. In the nonrelativistic regime, it can easily be shown that the addition of a second electric field does not alter the qualitative nature of the motion. Regardless of the polarization, frequency, and phase of the second field, the motion in the frame rotating at the Larmor frequency appears still as driven harmonic oscillations; only the number of the driving terms doubles. The matter is not that simple in the relativistic regime. Consider the motion in the presence of a uniform magnetic field and two circularly polarized electric fields with the second electric field given by

$$\vec{E'}=E_0'\cos(w't+\theta)\hat{e}_x+E_0'\sin(w't+\theta)\hat{e}_y, \tag{28}$$

where θ represents the phase difference between the two electric fields. Unless the frequencies of the two electric fields are equal, it is not possible to find a rotating frame in which the Hamiltonian is time independent. It therefore appears unlikely that relativistic cyclotron motion in the presence of two or more electric fields is integrable. A definite statement about nonintegrability in this case, however, can be made only after further computation and analysis are performed.

III. SUMMARY AND DISCUSSION

We have shown that nonrelativistic cyclotron motion is regular regardless of the polarization and the number of the electric fields, while relativistic cyclotron motion is integrable probably only when there is one circularly polarized electric field. For the case of a circularly polarized electric field, the motion is essentially equivalent to that in the presence of static electric and magnetic fields, when viewed in the frame rotating at the frequency of the electric field. If the electric field is not circularly polarized, however, the electric field is still time dependent in that rotating frame and, for that matter, in any other rotating frame. Nevertheless, if the motion remains nonrelativistic, it is advantageous to consider the motion in the frame rotating at Larmor frequency. In this rotating frame, the electric field is still time dependent, but the motion appears simply as driven harmonic oscillations. If the motion is relativistic, however, the precessing frequency of a charged particle in a uniform magnetic field is no longer constant. It is not possible to find a rotating frame in which the motion looks ‘‘simple,’’ except of course when the field is circularly polarized.

It is evident that integrability of the cyclotron motion can be proved most readily if an appropriate rotating frame of reference is found. From a fundamental viewpoint, such proof is valid because the transformation to a rotating frame of reference is canonical. We also mention in passing that our treatment of the cyclotron motion is approximate because the wave nature of the charged particles is not considered.

Finally, it should be mentioned that the question concerning the effect of the polarization of the electric field upon the dynamics of charged particles is of much current interest in the field of nonlinear dynamics, especially in studies of multiphoton ionization of the hydrogen atom. The classical motion of the electron in the hydrogen in the presence of an oscillating electric field is nonintegrable, even if relativistic effects are neglected, regardless of the polarization of the field. It has been observed, however, that the critical field above which the electron displays chaotic behavior and at which ionization begins to take place is higher when the field is circularly polarized than when it is linearly polarized [12–15]. Comparing the electron motion in the hydrogen and the cyclotron motion being considered here, while the cyclotron motion can exhibit chaos only when the motion becomes relativistic and occurs in a noncircularly polarized electric field, the electron motion in the hydrogen is nonintegrable already in the nonrelativistic regime for any polarization. Yet it is interesting to note that, in either case, a circularly polarized electric field provides a condition more strongly resistant to nonintegrability and chaos.

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